Capacity Bounds for a Class of Diamond Networks with Conferencing Relays

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Joint work with **Wanyao Zhao** and **Prof. Ashish Khisti**, 2015

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System Setup

Channel Model

This is the general setup of a diamond channel:

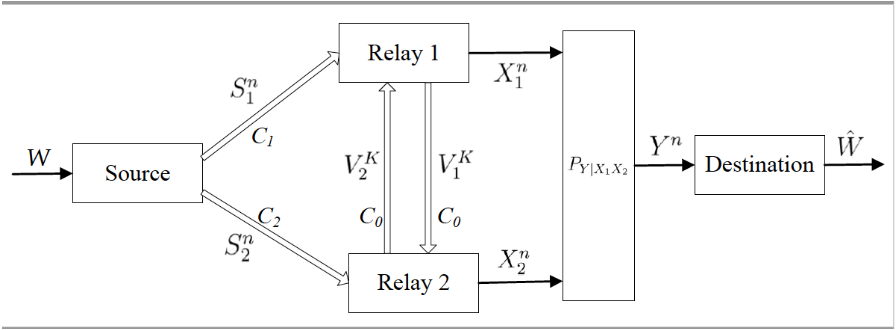

Figure : Diamond network with relay conferencing

Channel Model Continued

The particular channel being studied has three parts:

1. The broadcasting portion
   - Modelled by two noise-less bitpipes, each of capacities C_1 and C_2 from the source to relays 1 and 2, respectively.
2. The conferencing portion
   - Modelled by two identical noise-less bitpipes of capacity C_0 from one relay to the other
3. The multiple-access portion
   - General scenario considered: $P_{Y|X_1,X_2}$

We are interested in studying the capacity of this channel and look at some special cases of the MAC.

Problem Definition

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- The source encoder maps message $W \in [1 : 2^{nR}]$ onto S_1^n and S_2^n and transmits through the noiseless bit-pipes to relays 1 and 2, respectively, i.e.

$$S_1^n = f_{0,1}(W), \quad S_2^n = f_{0,2}(W). \tag{1}$$

  S_1^n and S_2^n are such that $H(S_1^n) \le nC_1$ and $H(S_2^n) \le nC_2$.

- After receiving S_i^n, two relays can communicate through K rounds in a round-robin fashion. At round k, relay i sends $V_{i,k}$ to the other relay based on S_i^n and message from the other relay in previous rounds, i.e.

$$V_{1,k} = f_{1,2}(S_1^n, V_2^{k-1}), \quad V_{2,k} = f_{2,1}(S_2^n, V_1^{k-1}) \tag{2}$$

  for $k = 1, 2, ..., K$ and $V_{i,0} = 1$ for $i = 1, 2$. V_i^K satisfies the constraint $H(V_i^K) \le nC_0$ for $i = 1, 2$.

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Problem Definition

- Then relay i maps its received signal S_i^n from the transmitter and $V_{\{1,2\}\setminus i}^K$ from the other relay to X_i^n as the input to the MAC for $i = 1, 2$, i.e.

$$X_1^n = f_{1,3}(S_1^n, V_2^K), \quad X_2^n = f_{2,3}(S_2^n, V_1^K). \tag{3}$$

- The MAC is characterized by its input alphabet $\mathcal{X}_1$, $\mathcal{X}_2$, output alphabet $\mathcal{Y}$ and transition probability $p(y|x_1, x_2)$. The receiver decodes an estimate $\hat{W} = g(Y^n)$ of W .

Definition

The capacity of the diamond network considered is defined as the maximum R such that for any $\epsilon > 0$ there exists $\{f_{i,0}, f_{i,\{1,2\}\setminus i}, f_{i,3}, g\}, i = 1, 2$ and n sufficiently large such that $Pr(W \neq \hat{W}) \leq \epsilon$.

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Capacity Bounds

Cutset Bound

The cut-set bound of the diamond channel is as follows:

$$R \leq \max_{p(x_1,x_2)} \min \begin{cases} C_1 + C_2 \\ I(X_1, X_2; Y) \\ C_1 + C_0 + I(X_2; Y|X_1) \\ C_2 + C_0 + I(X_1; Y|X_2). \end{cases} \tag{4}$$

The cut-set bound does not fully capture the trade-off between sending independent messages at the encoder, represented by the $C_1 + C_2$ term, and the full cooperation at the MAC, represented by the $I(X_1, X_2; Y)$ term. A tighter upper bound can be derived.

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Improved Upper Bound

Theorem [Zhao-Ding-Khisti '15]

An upper bound of the diamond network with conference is

$$R \le \max_{p(x_1,x_2)} \min_{p(u|x_1,x_2,y)} \beta,$$

where

$$\beta = \min \begin{cases} C_1 + C_2 \\ I(X_1, X_2; Y) \\ C_1 + C_0 + I(X_2; Y|X_1) \\ C_2 + C_0 + I(X_1; Y|X_2) \\ \frac{1}{2}(C_1 + C_2 + 2C_0 + I(X_1, X_2; Y|U) \\ \qquad +I(X_1; U|X_2) + I(X_2; U|X_1)). \end{cases} \tag{5}$$

with $|\mathcal{U}| \le |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| + 2.$

Proof Sketch

Starting from Fano's inequality, we have

$$nR \le I(W;Y^n) + n\epsilon \quad (6)$$
$$\cdots \quad (7)$$
$$\le nC_1 + nC_2 - I(S_1^n;S_2^n) + n\epsilon \quad (8)$$

- The $I(S_1^n;S_2^n) \ge 0$ term captures the trade-off!

Observe: $V_1^K = f(S_1^n, V_2^{K-1})$ and $V_2^{K-1} = f(S_2^n, V_1^{K-2})$
Therefore, V_1^K is a function of both S_1^n and S_2^n. Same for V_2^K.

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Proof Sketch

We need to find the relation between $I(S_1^n;S_2^n)$ and $I(X_1^n;X_2^n)$, as follows,

$$\begin{aligned} & I(X_1^n;X_2^n) & (9)\\ \le\ & I(S_1^n,V_2^K;S_2^n,V_1^K) & (10)\\ =\ & I(S_1^n;S_2^n)+I(V_2^K;S_2^n|S_1^n) & \\ & +I(V_1^K;S_1^n|S_2^n)+I(V_1^K;V_2^K|S_1^n,S_2^n), & (11) \end{aligned}$$

where (10) is due to the Markov chain $X_1^n \leftrightarrow (S_1^n,V_2^K) \leftrightarrow (S_2^n,V_1^K) \leftrightarrow X_2^n$. For the terms in (11),

$$I(V_1^K;S_1^n|S_2^n) = H(V_1^K|S_2^n) \quad (12)$$

$$I(V_2^K;S_2^n|S_1^n) = H(V_2^K|S_1^n) \quad (13)$$

$$I(V_1^K;V_2^K|S_1^n,S_2^n) = 0, \quad (14)$$

are true since V_1^K and V_2^K are both deterministic functions of (S_1^n,S_2^n).

Proof Sketch

Therefore, it follows that

$$\begin{aligned} & I(X_1^n;X_2^n) && (15)\\ \le & I(S_1^n;S_2^n)+H(V_2^K|S_1^n)+H(V_1^K|S_2^n) && (16)\\ \le & I(S_1^n;S_2^n)+H(V_2^K)+H(V_1^K) && (17)\\ \le & I(S_1^n;S_2^n)+2nC_0. && (18)\end{aligned}$$

Combining (8) with (18) we have

$$nR \le nC_1+nC_2+2nC_0-I(X_1^n;X_2^n)+n\epsilon. \quad (19)$$

Finally, we use techniques in [Bidokhti-Kramer '14] to single-letterize the bound in (19) to obtain that

$$\begin{aligned} 2R \le & C_1+C_2+2C_0+I(X_1,X_2;Y|U)\\ & +I(X_1;U|X_2)+I(X_2;U|X_1) \quad (20)\end{aligned}$$

for every auxiliary channel $p(u|x_1,x_2,y)$. Combining with the cut-set bound, we finished the proof.

A Lower Bound

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Theorem [Zhao-Ding-Khisti '15]

For diamond network with conferencing relays, rate R is achievable if for some pmf $p(u,x_1,x_2,y)=p(u,x_1,x_2)p(y|x_1,x_2)$ and $U\in\mathcal{U}$ with $|\mathcal{U}|\le\min\{|\mathcal{X}_1||\mathcal{X}_2|+2,|\mathcal{Y}|+4\}$, it satisfies that

$$R\le\min\begin{cases}C_0+C_2+I(X_1;Y|X_2,U)\\ C_0+C_1+I(X_2;Y|X_1,U)\\ C_1+C_2-I(X_1;X_2|U)\\ I(X_1,X_2;Y)\\ \frac{1}{2}(C_1+C_2+2C_0-I(X_1;X_2|U)\\ \quad +I(X_1,X_2;Y|U)).\end{cases}\tag{21}$$

Proof sketch: Marton's coding with rate-splitting. Codebook is taken from [Bidokhti-Kramer '14] and is added extra list of messages for the relays to transmit to each other through conferencing links. Error analysis is also an extension of those done in [Bidokhti-Kramer '14].

A LOWER BOUND

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$$R \leq \min \begin{cases} C_0 + C_2 + I(X_1; Y|X_2, U) \\ C_0 + C_1 + I(X_2; Y|X_1, U) \\ C_1 + C_2 - I(X_1; X_2|U) \\ I(X_1, X_2; Y) \\ \frac{1}{2}(C_1 + C_2 + 2C_0 - I(X_1; X_2|U) \\ \quad + I(X_1, X_2; Y|U)). \end{cases} \tag{21}$$

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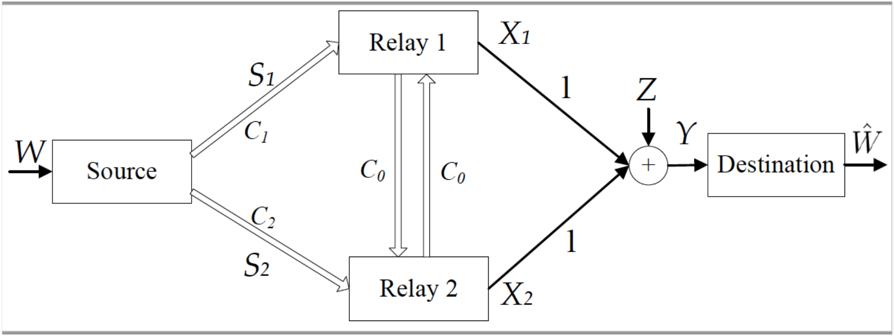
Case Studies

Symmetric Gaussian Diamond Channel (SGDC)



$C_1 = C_2 = C$ and Gaussian MAC. The output of the Gaussian MAC is

$$Y = X_1 + X_2 + Z, \tag{22}$$

where $Z \sim \mathcal{N}(0,1)$. Both X_1 and X_2 have average power constraint $\frac{1}{n}\sum_{i=1}^{n} E[X_{j,i}^2] \leq P$ for $j = 1, 2$.

Upper-Bounds on SGDC

Applying theorem 1 and techniques similar to [Kang-Liu '11], we arrive at the following results

Theorem [Zhao-Ding-Khisti '15]

$$C^{+} = \max\{\max_{\rho \leq \rho*} S_1(\rho), \max_{\rho* \leq \rho \leq 1} S_2(\rho)\}. \tag{23}$$

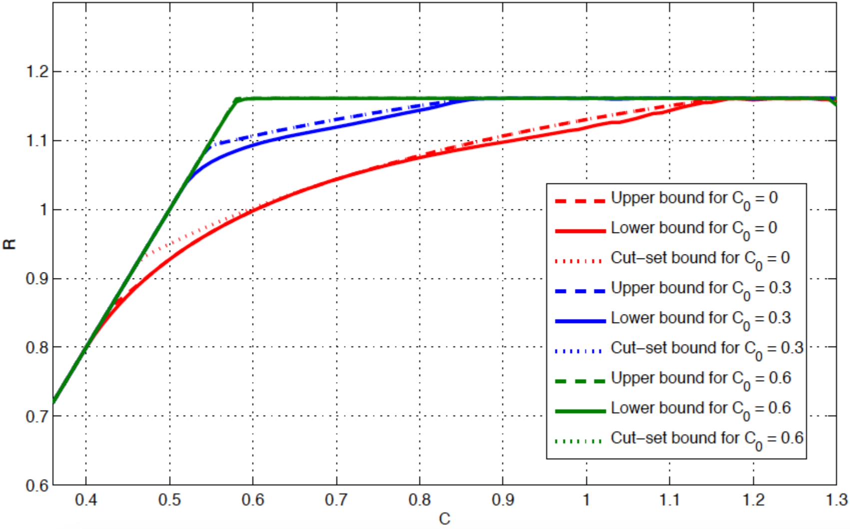
where

$$S_1(\rho) = \min \begin{cases} 2C \\ \frac{1}{2}\log(1+2(1+\rho)P) \\ C + C_0 + \frac{1}{2}\log(1+(1-\rho^2)P) \\ C + C_0 + \frac{1}{4}\log(1+2(1+\rho)P) \\ \qquad - \frac{1}{2}\log\frac{1}{1-\rho^2} \end{cases}$$

$$S_2(\rho) = \min \begin{cases} 2C \\ \frac{1}{2}\log(1+2(1+\rho)P) \\ C + C_0 + \frac{1}{2}\log(1+(1-\rho^2)P) \end{cases}$$

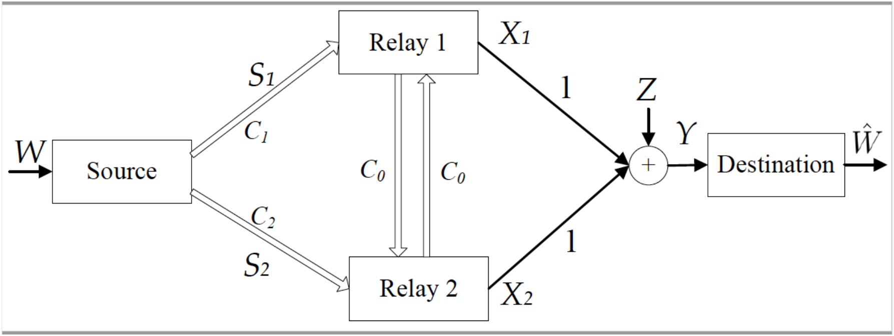
$$\rho* = \sqrt{1+\frac{1}{4P^2}} - \frac{1}{2P}. \tag{24}$$

SGDC


1.2
1.1
1
0.9
0.8
0.7
0.6
R
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
C
Upper bound for $C_0 = 0$
Lower bound for $C_0 = 0$
Cut–set bound for $C_0 = 0$
Upper bound for $C_0 = 0.3$
Lower bound for $C_0 = 0.3$
Cut–set bound for $C_0 = 0.3$
Upper bound for $C_0 = 0.6$
Lower bound for $C_0 = 0.6$
Cut–set bound for $C_0 = 0.6$

Minimum C_0 required

Research question: we would like to use the conferencing links as "helpers" to allow the broadcast bitpipes to transmit independent messages and yet have the MAC achieving full cooperative potential. What is the minimum C_0 required?



i.e. $C_0(C)$

Minimum C_0 required

- For $C < \frac{1}{4}\log(1+4P)$, C_0 is not possible
- For $C \geq \frac{1}{2}\log(1+4P)$, $C_0 = 0$

Interesting regime is $C \in [\frac{1}{4}\log(1+4P), \frac{1}{2}\log(1+4P))$

Theorem [Zhao-Ding-Khisti '15]

Given the capacity of the source-to-relay bit-pipe links $C \in [\frac{1}{4}\log(1+4P), \frac{1}{2}\log(1+4P)]$, the cut-set bound $R \leq \frac{1}{2}\log(1+4P)$ can be achieved if and only if $C_0 \geq \frac{1}{2}\log(1+4P) - C$.

PROOF SKETCH

Achievability is obvious. For the converse, note that for (23), in the case of $\frac{1}{4}\log(1+4P) \leq C < \frac{1}{2}\log(1+4P)$, the constraint $2C$ for both $S_1(\rho)$ and $S_2(\rho)$, respectively, is inactive.
Now, in this regime, observe that

$$\begin{aligned}\max_{\rho\leq\rho*} S_1(\rho) &\leq \max_{\rho\leq\rho*}\frac{1}{2}\log(1+2(1+\rho)P)\\ &< \frac{1}{2}\log(1+4P), \qquad (25)\end{aligned}$$

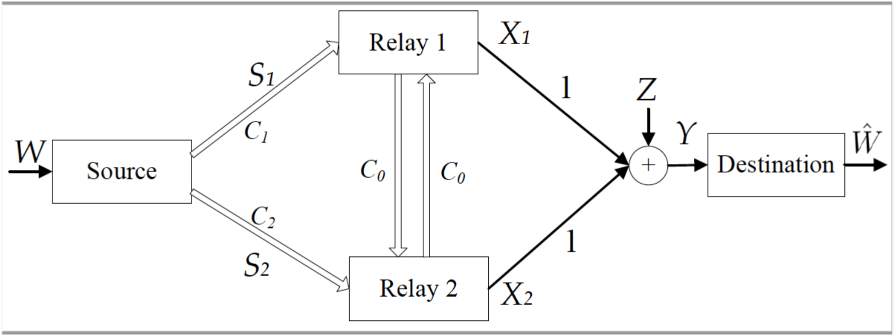
thus we have $C^+ = \frac{1}{2}\log(1+4P)$ if and only if $\max_{\rho*\leq\rho\leq 1} S_2(\rho) = \frac{1}{2}\log(1+4P)$. Since for $\rho* < \rho < 1$ we have

$$S_2(\rho) \leq \frac{1}{2}\log(1+2(1+\rho)P) < \frac{1}{2}\log(1+4P), \qquad (26)$$

$\max_{\rho*\leq\rho\leq 1} S_2(\rho) = \frac{1}{2}\log(1+4P)$ is satisfied if and only if $S_2(1) = \frac{1}{2}\log(1+4P)$, which implies

$$C + C_0 \geq \frac{1}{2}\log(1+4P), \qquad (27)$$

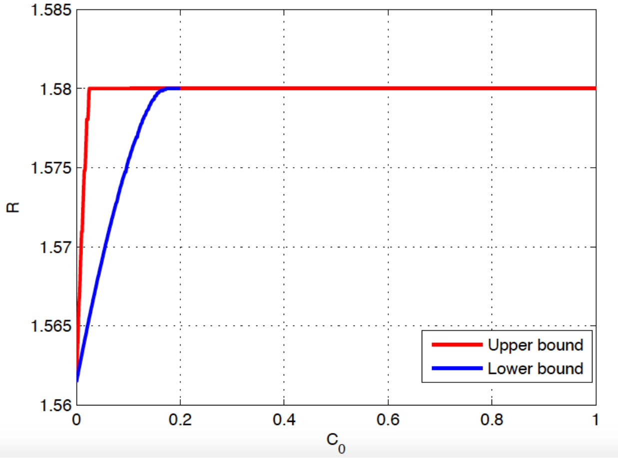
Result Not Trivial—Binary Adder Channel



We let $\mathcal{Z} = 0$, $C_1 = C_2 = C$, input alphabet is $\mathcal{X} = \{0, 1\}$ and the output alphabet is $\mathcal{Y} = \{0, 1, 2\}$, with $Y = X_1 + X_2$.

BAC

Results on bounds can only be obtained numerically, and they are shown below for $C = \log(3)/2 = 0.79$, which is half of the full cooperative potential $\log(3) = 1.58$.



BAC

- Fig. 23 shows that the cut-set bound 1.58 for $C = 0.79$ is achievable if $C_0 \geq 0.174$ and only if $C_0 \geq 0.025$.
- It can be observed that unlike the Gaussian case, the minimum C_0 needed is much less than the difference between the cut-set bound 1.58 and $C = 0.79$, which is 0.79.

Why? (Long-story short)

- For the Gaussian case, in order to obtain the optimal $p(x_1, x_2)^*$, we need to have $X_1 = X_2$. Since we also need to form Markov chain $X_1 \leftrightarrow U \leftrightarrow X_2$, U has to be picked as a deterministic function of X_1 (or X_2). However, this is not true in general.
- In BAC, for example, with $p(u) = [\frac{1}{2}, \frac{1}{2}]$, $p(x_1|u) = p(x_2|u) = [\alpha, 1-\alpha; \beta, 1-\beta]$ with $\alpha = \frac{1}{2} - \frac{\sqrt{3}}{6}, \beta = \frac{1}{2} + \frac{\sqrt{3}}{6}$, we have $\sum_{u \in \mathcal{U}} p(u, x_1, x_2) = [\frac{1}{3}, \frac{1}{6}; \frac{1}{6}, \frac{1}{3}]$ and $I(X_1; Y|X_2, U) = I(X_2; Y|X_1, U) = 0.7440$, $\frac{1}{2} I(X_1, X_2; Y|U) = 0.5774$, hence when $C_0 \geq 0.2152$, R_{CS} of 1.5850 can be achieved given $C = 0.7925 = \frac{1}{2} R_{CS}$.

Conclusion

- Effects of relay conferencing on a class of diamond networks are examined
- Tighter upper bound and a lower bound on its capacity is presented
- There is no clever manipulation of conferencing links to achieve full MAC cooperation with individual source-to-relay messages for Gaussian MACs
- However, this is not true in general as seen from Binary Adder MAC